STAT 135 Lab 10
Two-Way ANOVA, Randomized Block Design and Friedman’s Test

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Two-way ANOVA
Let’s now imagine a dataset for which our response variable, $Y$, may be influenced by two factors, each of which are potential sources of variation.

- The first factor has $I$ levels
- The second factor has $J$ levels

Each combination $(i, j)$ defines a treatment pairing. There are a total of $IJ$ treatment pairs.

We assume that we have $K > 1$ replicates for each treatment combination $(i, j)$. 

**Two-way ANOVA**
Two-way ANOVA

The idea is that we want to simultaneously examine the effects of the two treatments/factors, and their interaction, on the response.

So we actually conduct 3 $F$-tests (one for each factor and one for their interaction)

- To test if factor $A$ has an effect:
  - compare the variability between the groups of factor $A$ to the “within” variability

- To test if factor $B$ has an effect:
  - compare the variability between the groups of factor $B$ to the “within” variability

- To test if there is an interaction effect between $A$ and $B$:
  - compare the variability between each combination of the groups of factor $A$ and of factor $B$ to the “within” variability
Two-way ANOVA

Suppose that a reporter wants to know if the salary at a small company varies according to age and gender.

- The response variable is salary (multiples of $1000)
- The two factors being studied are age and gender

Suppose that we have two people who fit into each category (so we have 2 replicates)

<table>
<thead>
<tr>
<th></th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>age less than 40</td>
<td>71, 75</td>
<td>67, 70</td>
</tr>
<tr>
<td>age 40 to 55</td>
<td>77, 81</td>
<td>79, 82</td>
</tr>
<tr>
<td>age above 55</td>
<td>82, 80</td>
<td>87, 77</td>
</tr>
</tbody>
</table>

\[ I = 3 \quad J = 2 \quad K = 2 \]
Two-way ANOVA

Let’s introduce some notation:

- $Y_{ijk}$ is the $k^{th}$ observation in the $i^{th}$ row and $j^{th}$ column.
- $\bar{Y}_{i.}$ is the mean of all observations in the $i^{th}$ row.
- $\bar{Y}_{.j}$ is the mean of all observations in the $j^{th}$ column.
- $\bar{Y}_{ij.}$ is the mean of all observations in the $(i,j)^{th}$ cell.
- $\bar{Y}_{...}$ is the mean of all observations

where a $\cdot$ in the subscript indicates what we are averaging over.
Two-way ANOVA

Note that we can represent the two-way ANOVA as a simple model by modeling the response $Y_{ijk}$ as

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \delta_{ij} + \epsilon_{ijk}$$

- $\mu$ is the **global mean**
- $\alpha_i$ describes how level $i$ of the **row factor** affects the response
- $\beta_j$ describes how level $j$ of the **column factor** affects the response
- $\delta_{ij}$ describes how level $i$ of the row factor and level $j$ of the column factor **interact** to affect the response
- $\epsilon_{ijk}$ describes the **random errors for each observation** and each $\epsilon_{ijk}$ independently satisfies

$$\epsilon_{ijk} \sim N(0, \sigma^2)$$
Two-way ANOVA

\[ Y_{ijk} = \mu + \alpha_i + \beta_j + \delta_{ij} + \epsilon_{ijk} \]

Note that \( \epsilon_{ijk} \sim N(0, \sigma^2) \) implies that

\[ E(Y_{ijk}) = \mu + \alpha_i + \beta_j + \delta_{ij} \]

The parameters also satisfy the following constraints:

\[
\begin{align*}
\sum_{i=1}^{I} \alpha_i &= 0 \\
\sum_{j=1}^{J} \beta_j &= 0 \\
\sum_{i=1}^{I} \delta_{ij} &= \sum_{j=1}^{J} \delta_{ij} = 0
\end{align*}
\]
Two-way ANOVA

\[ Y_{ijk} = \mu + \alpha_i + \beta_j + \delta_{ij} + \epsilon_{ijk} \]

We can calculate maximum likelihood estimates of each of these parameters:

\[ \hat{\mu} = \bar{Y} \ldots \quad \text{global mean} \]

\[ \hat{\alpha}_i = \bar{Y}_{i.} - \bar{Y} \ldots \quad \text{differential effects for the row factor} \]

\[ \hat{\beta}_j = \bar{Y}_{.j} - \bar{Y} \ldots \quad \text{differential effects for the column factor} \]

\[ \hat{\delta}_{ij} = \bar{Y}_{ij} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j \quad \text{interaction effect} \]
Two-way ANOVA

For our example:

<table>
<thead>
<tr>
<th>Age Group</th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>age less than 40</td>
<td>71, 75</td>
<td>67, 70</td>
</tr>
<tr>
<td>age 40 to 55</td>
<td>77, 81</td>
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</tr>
<tr>
<td>age above 55</td>
<td>82, 80</td>
<td>87, 77</td>
</tr>
</tbody>
</table>

The mean over each age group is given by:

\[
\bar{Y}_{1..} = \frac{71 + 75 + 67 + 70}{2 \times 2} = 70.75
\]

\[
\bar{Y}_{2..} = \frac{77 + 81 + 79 + 82}{2 \times 2} = 79.75
\]

\[
\bar{Y}_{3..} = \frac{82 + 80 + 87 + 77}{2 \times 2} = 81.50
\]
Two-way ANOVA

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<thead>
<tr>
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</tr>
<tr>
<td>above 55</td>
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<td>87, 77</td>
</tr>
</tbody>
</table>

The mean over each gender is given by:

\[
\bar{Y}_1 = \frac{71 + 75 + 77 + 81 + 82 + 80}{3 \times 2} = \frac{480}{6} = 77.67
\]

\[
\bar{Y}_2 = \frac{67 + 70 + 79 + 82 + 87 + 77}{3 \times 2} = \frac{440}{6} = 77.00
\]
Two-way ANOVA

For our example:

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<td>87, 77</td>
</tr>
</tbody>
</table>

The mean over each gender/age combination is given by:

\[
\bar{Y}_{11} = \frac{71 + 75}{2} = 73.0
\]

\[
\bar{Y}_{12} = \frac{67 + 70}{2} = 68.5
\]

\[
\bar{Y}_{21} = \frac{77 + 81}{2} = 79.0
\]

\[
\bar{Y}_{22} = \frac{79 + 82}{2} = 80.5
\]

\[
\bar{Y}_{31} = \frac{82 + 80}{2} = 81.0
\]

\[
\bar{Y}_{32} = \frac{87 + 77}{2} = 83.0
\]
Two-way ANOVA

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</tr>
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</table>

The global mean is given by:

$$\overline{Y} = \frac{71 + 75 + 67 + 77 + 81 + 79 + 82 + 82 + 80 + 87 + 77}{3 \times 2 \times 2} = 77.33$$
Two-way ANOVA

Like one-way ANOVA, the two-way ANOVA is conducted by comparing various sum of squares. Recall that for one-way ANOVA we had

\[ SS_T = SS_B + SS_W \]

where \( SS_B \) was the between groups sum of squares and \( SS_W \) was the within groups sum of squares.
Two-way ANOVA

For two-way ANOVA, the total variability can be described by the total sum of squares

$$SS_T = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (Y_{ijk} - \bar{Y}_{..})^2$$

and we will see that the total variability can be split up into the within-cell variability and the between-group variability, where

- The within-cell variability (often referred to as errors or residuals) is defined by $SS_E$ (analog of $SS_W$)

$$SS_E = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (Y_{ijk} - \bar{Y}_{ij.})^2$$
Two-way ANOVA

- The **between-cell variability** can be split up into three components (analog of $SS_B$)
  - $SS_A$: the sum of squares for the row factor
    \[
    SS_A = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (\bar{Y}_{i..} - \bar{Y}...)^2 = JK \sum_{i=1}^{I} (\bar{Y}_{i..} - \bar{Y}...)^2
    \]
  - $SS_B$: the sum of squares for the column factor
    \[
    SS_B = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (\bar{Y}_{.j} - \bar{Y}...)^2 = IK \sum_{j=1}^{J} (\bar{Y}_{.j} - \bar{Y}...)^2
    \]
  - $SS_{AB}$: the sum of squares for the interaction between the row and column factor
    \[
    SS_{AB} = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (\bar{Y}_{ij} - \bar{Y}_{i..} - \bar{Y}_{.j} + \bar{Y}...)^2
    \]
    \[
    = K \sum_{i=1}^{I} \sum_{j=1}^{J} (\bar{Y}_{ij} - \bar{Y}_{i..} - \bar{Y}_{.j} + \bar{Y}...)^2
    \]
Two-way ANOVA

One way to graphically determine if there is an interaction effect is to use an interaction plot.

- $y$-axis is the response level
- $x$-axis is the levels of one of the factors
- There is a different curve for each level of the other factor.

If the curves appear to be more or less parallel, then we conclude that there is unlikely to be an interaction effect.
Two-way ANOVA

There appears to be no interaction for the older age groups with gender, but a bit of an interaction within the younger age groups with gender. We will see that this is not enough to conclude that there is a significant interaction over all levels.
Two-way ANOVA

The sum of squares satisfy

$$SS_T = SS_A + SS_B + SS_{AB} + SS_E$$

and if we assume that $\epsilon_{ijk} \sim N(0, \sigma^2)$, then

- $SS_E/\sigma^2 \sim \chi^2_{IJ(K-1)}$
- $SS_A/\sigma^2 \sim \chi^2_{I-1}$
- $SS_B/\sigma^2 \sim \chi^2_{J-1}$
- $SS_{AB}/\sigma^2 \sim \chi^2_{(I-1)(J-1)}$
## Two-way ANOVA

The two-way ANOVA table is thus given by

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>A (between)</td>
<td>$I - 1$</td>
<td>$SS_A$</td>
<td>$MS_A$</td>
<td>$MS_A/MS_E$</td>
</tr>
<tr>
<td>B (between)</td>
<td>$J - 1$</td>
<td>$SS_B$</td>
<td>$MS_B$</td>
<td>$MS_B/MS_E$</td>
</tr>
<tr>
<td>AB (between)</td>
<td>$(I - 1)(J - 1)$</td>
<td>$SS_{AB}$</td>
<td>$MS_{AB}$</td>
<td>$MS_{AB}/MS_E$</td>
</tr>
<tr>
<td>Error (within)</td>
<td>$IJ(K - 1)$</td>
<td>$SS_E$</td>
<td>$MS_E$</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>$IJK - 1$</td>
<td>$SS_T$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where

$$MS = \frac{SS}{df}$$
Two-way ANOVA

The \( p \)-value for testing the null hypothesis that there is no difference between the levels of factor \( A \) is given by

\[
P(F_{I-1, IJ(K-1)} \geq MS_A/MS_E)
\]

The \( p \)-value for testing the null hypothesis that there is no difference between the levels of factor \( B \) is given by

\[
P(F_{J-1, IJ(K-1)} \geq MS_B/MS_E)
\]

The \( p \)-value for testing the null hypothesis that there is no interaction between \( A \) and \( B \) is given by

\[
P(F_{(I-1)(J-1), IJ(K-1)} \geq MS_{AB}/MS_E)
\]
Two-way ANOVA

For our example,

<table>
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\[ SS_{Age} = 266.17 \]
\[ SS_{Gender} = 1.33 \]
\[ SS_{Age \times Gender} = 22.17 \]
\[ SS_{Error} = 77.00 \]
Two-way ANOVA

For our example:

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gender</td>
<td>1</td>
<td>1.13</td>
<td>1.13</td>
<td>0.10</td>
</tr>
<tr>
<td>Age</td>
<td>2</td>
<td>226.17</td>
<td>133.08</td>
<td>10.37</td>
</tr>
<tr>
<td>Age × Gender</td>
<td>2</td>
<td>22.17</td>
<td>11.08</td>
<td>0.86</td>
</tr>
<tr>
<td>Error</td>
<td>6</td>
<td>77.00</td>
<td>12.83</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>11</td>
<td>366.67</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

and our $p$-values are:

Gender effect: $P(F_{1,6} \geq 0.10) = 0.76$

Age effect: $P(F_{2,6} \geq 10.37) = 0.011$

Interaction effect: $P(F_{2,6} \geq 0.86) = 0.47$

So we conclude that there is a difference in salary between the different age groups but not for the different genders. There also does not appear to be a significant interaction effect.
Exercise
Exercise: Two-way ANOVA

Do the salary example in R, and report your results in a .pdf format. Options include (but are not limited to):

- Using knitr (.Rnw file) – my preferred method (allows for easy incorporation of LaTeX, R code and figures)
- Using R Markdown (.Rmd file)
- Using IPython notebook
Randomized Block Design
Randomized Block Design

Suppose we have $I$ treatments (e.g. fertilizers, diets) that we want to try on each of $I$ subjects (e.g. plots of land, people).

The $I$ blocks for a subject might be

- physical partitions of the plot of land where a different fertilizer is applied to each of the blocks
- stretches of time in each of which the same subject is put on a different diet.
Randomized Block Design

How is this different from one-way ANOVA?
- One-way ANOVA: we have a different group of $J$ subjects for each of the $I$ treatments
- Randomized Block Design: we have the same group of $J$ subjects for each of the $I$ treatments

How is this different from two-way ANOVA?
- Two-way ANOVA: Subjects are considered replicates within factors
- Randomized Block Design: Subjects are themselves a factor
Randomized Block Design

The observation in the \((i, j)\)th block, \(Y_{ij}\), is modeled by

\[
Y_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij}
\]

- \(\mu\) is the overall mean parameter
- \(\alpha_i\) is the differential effect of the \(i\)th treatment.
  - Assume \(\sum_{i=1}^{I} \alpha_i = 0\).
- \(\beta_j\) is the differential effect of the \(j\)th subject.
  - Assume \(\sum_{j=1}^{J} \beta_j = 0\).
- \(\epsilon_{ij}\) is the random error for the \(i\)th treatment on the \(j\)th subject.
  - Assume \(\epsilon_{ij} \sim \text{IID} N(0, \sigma^2)\).
Randomized Block Design

The total amount of variation is given by:

$$SS_T = \sum_{i=1}^{I} \sum_{j=1}^{J} (Y_{ij} - \bar{Y})^2$$

The variation explained by the treatment differential effect is given by:

$$SS_A = J \sum_{i=1}^{I} (\bar{Y}_{i} - \bar{Y})^2$$

The variation explained by the subject differential effect is given by:

$$SS_B = I \sum_{j=1}^{J} (\bar{Y}_{j} - \bar{Y})^2$$

The variation not explained by the model is given by:

$$SS_{AB} = \sum_{i=1}^{I} \sum_{j=1}^{J} (Y_{ij} - \bar{Y}_{i} - \bar{Y}_{j} + \bar{Y})^2$$
Randomized Block Design

As usual, we have

\[ SS_T = SS_A + SS_B + SS_{AB} \]

and that

\[ MS_A = \frac{SS_A}{I - 1} \quad MS_B = \frac{SS_B}{J - 1} \quad MS_{AB} = \frac{SS_{AB}}{(I - 1)(J - 1)} \]

To test the null hypothesis that there is no treatment effect:

\[ H_0 : \alpha_i = 0 \quad \forall i \]

our \( p \)-value is given by

\[ P \left( F_{I-1,(I-1)(J-1)} \geq \frac{MS_A}{MS_{AB}} \right) \]
Exercise
Exercise: Randomized block design (Rice 12.3 Example A)

Let’s consider an experimental study of drugs to relieve itching.

- 5 drugs were compared to a placebo and no drug (7 treatments).
- 10 volunteers male subjects aged 20-30.
- Each volunteer underwent one treatment per day, and the time-order was randomized.
- Individuals (rather than treatments) are the “blocks”.
- The subjects were given a drug (or placebo) intravenously, and itching was induced on their forearms.
- The subjects recorded the duration of the itching.
Exercise: Randomized block design (Rice 12.3 Example A)

The following table recorded the durations of the itching (in seconds) for the first 5 subjects.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>BG</td>
<td>174</td>
<td>263</td>
<td>105</td>
<td>199</td>
<td>141</td>
<td>108</td>
<td>141</td>
</tr>
<tr>
<td>JF</td>
<td>224</td>
<td>213</td>
<td>103</td>
<td>143</td>
<td>168</td>
<td>341</td>
<td>184</td>
</tr>
<tr>
<td>BS</td>
<td>260</td>
<td>231</td>
<td>145</td>
<td>113</td>
<td>78</td>
<td>159</td>
<td>125</td>
</tr>
<tr>
<td>SI</td>
<td>225</td>
<td>291</td>
<td>103</td>
<td>225</td>
<td>164</td>
<td>135</td>
<td>227</td>
</tr>
<tr>
<td>BW</td>
<td>165</td>
<td>168</td>
<td>144</td>
<td>176</td>
<td>127</td>
<td>239</td>
<td>194</td>
</tr>
</tbody>
</table>

Test the null hypothesis that there is no difference in means between the different treatments.
Friedman’s Test (non-parametric version of the Randomized block design)
1. Calculate the ranks of each treatment within each of the $J$ blocks (rather than overall). $R_{ij}$ is the rank of the $i$th treatment for the $j$th subject for the $j$th subject.

2. Compute

$$Q = \frac{12}{I(I+1)} SS_A = \frac{12J}{I(I+1)} \sum_{i=1}^{I} (\bar{R}_i - \bar{R})^2 \sim \chi_{I-1}^2$$

To test the null hypothesis that there is no treatment effect, the p-value is given by

$$P(\chi_{I-1}^2 \geq Q)$$
Exercise
Exercise: Friedman’s test (Rice 12.4 Example A)

Do the previous example using a non-parametric method.