STAT 135 Lab 10
Two-Way ANOVA, Randomized Block Design and Friedman's Test

Rebecca Barter

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Two-way ANOVA

Two-way ANOVA

Let's now imagine a dataset for which our response variable, Y, may be influenced by two **factors**, each of which are potential sources of variation.

- The first factor has I levels
- The second factor has J levels

Each combination (i, j) defines a **treatment** pairing. There are a total of IJ treatment pairs.

We assume that we have $K > 1$ replicates for each treatment combination (i, j).

Two-way ANOVA

The idea is that we want to simultaneously examine the effects of the two treatments/factors, and their interaction, on the response.

So we actually conduct 3 F-tests (one for each factor and one for their interaction)

- To test if factor A has an effect:
  - compare the variability between the groups of factor A to the "within" variability
- To test if factor B has an effect:
  - compare the variability between the groups of factor B to the "within" variability
- To test if there is an interaction effect between A and B:
  - compare the variability between each combination of the groups of factor A and of factor B to the "within" variability

Two-way ANOVA

Suppose that a reporter wants to know if the salary at a small company varies according to age and gender.

- The response variable is **salary** (multiples of \$1000)
- The two factors being studied are **age** and **gender**

Suppose that we have two people who fit into each category (so we have 2 replicates)

	Male	Female
age less than 40	71, 75	67, 70
age 40 to 55	77, 81	79, 82
age above 55	82, 80	87, 77

$$I = 3 \qquad J = 2 \qquad K = 2$$

Two-way ANOVA

Let's introduce some notation:

- Y_{ijk} is the k^{th} observation in the i^{th} row and j^{th} column.
- $\overline{Y}_{i..}$ is the mean of all observations in the i^{th} row
- $\overline{Y}_{.j.}$ is the mean of all observations in the j^{th} column
- $\overline{Y}_{ij.}$ is the mean of all observations in the $(i,j)^{th}$ cell
- $\overline{Y}_{...}$ is the mean of all observations

where a $\cdot$ in the subscript indicates what we are averaging over.

Two-way ANOVA

Note that we can represent the two-way ANOVA as a simple model by modeling the response Y_{ijk} as

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \delta_{ij} + \epsilon_{ijk}$$

- $\boldsymbol{\mu}$ is the **global mean**
- $\boldsymbol{\alpha}_i$ describes how level i of the **row factor** affects the response
- $\boldsymbol{\beta}_j$ describes how level j of the **column factor** affects the response
- $\boldsymbol{\delta}_{ij}$ describes how level i of the row factor and level j of the column factor **interact** to affect the response
- $\boldsymbol{\epsilon}_{ijk}$ describes the **random errors for each observation** and each ϵ_{ijk} independently satisfies

$$\epsilon_{ijk} \sim N(0, \sigma^2)$$

Two-way ANOVA

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \delta_{ij} + \epsilon_{ijk}$$

Note that $\epsilon_{ijk} \sim N(0, \sigma^2)$ implies that

$$E(Y_{ijk}) = \mu + \alpha_i + \beta_j + \delta_{ij}$$

The parameters also satisfy the following constraints:

$$\sum_{i=1}^{I} \alpha_i = 0$$

$$\sum_{j=1}^{J} \beta_j = 0$$

$$\sum_{i=1}^{I} \delta_{ij} = \sum_{j=1}^{J} \delta_{ij} = 0$$

Two-way ANOVA

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \delta_{ij} + \epsilon_{ijk}$$

We can calculate maximum likelihood estimates of each of these parameters:

$\hat{\mu} = \overline{Y}_{...}$	global mean
$\hat{\alpha}_i = \overline{Y}_{i..} - \overline{Y}_{...}$	differential effects for the row factor
$\hat{\beta}_j = \overline{Y}_{.j.} - \overline{Y}_{...}$	differential effects for the column factor
$\hat{\delta}_{ij} = \overline{Y}_{ij.} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j$	interaction effect

Two-way ANOVA

For our example:

	Male	Female
age less than 40	71, 75	67, 70
age 40 to 55	77, 81	79, 82
age above 55	82, 80	87, 77

The mean over each age group is given by:

$$\overline{Y}_{1..} = \frac{71 + 75 + 67 + 70}{2 \times 2} = 70.75$$

$$\overline{Y}_{2..} = \frac{77 + 81 + 79 + 82}{2 \times 2} = 79.75$$

$$\overline{Y}_{3..} = \frac{82 + 80 + 87 + 77}{2 \times 2} = 81.50$$

Two-way ANOVA

For our example:

	Male	Female
age less than 40	71, 75	67, 70
age 40 to 55	77, 81	79, 82
age above 55	82, 80	87, 77

The mean over each gender is given by:

$$\overline{Y}_{\cdot 1 \cdot} = \frac{71 + 75 + 77 + 81 + 82 + 80}{3 \times 2} = 77.67$$

$$\overline{Y}_{\cdot 2 \cdot} = \frac{67 + 70 + 79 + 82 + 87 + 77}{3 \times 2} = 77.00$$

Two-way ANOVA

For our example:

	Male	Female
age less than 40	71, 75	67, 70
age 40 to 55	77, 81	79, 82
age above 55	82, 80	87, 77

The mean over each gender/age combination is given by:

$$\overline{Y}_{11\cdot} = \frac{71+75}{2} = 73.0 \qquad \overline{Y}_{12\cdot} = \frac{67+70}{2} = 68.5$$

$$\overline{Y}_{21\cdot} = \frac{77+81}{2} = 79.0 \qquad \overline{Y}_{22\cdot} = \frac{79+82}{2} = 80.5$$

$$\overline{Y}_{31\cdot} = \frac{82+80}{2} = 81.0 \qquad \overline{Y}_{32\cdot} = \frac{87+77}{2} 83.0$$

Two-way ANOVA

For our example:

	Male	Female
age less than 40	71, 75	67, 70
age 40 to 55	77, 81	79, 82
age above 55	82, 80	87, 77

The global mean is given by:

$$\overline{Y}_{...} = \frac{71 + 75 + 67 + 77 + 81 + 79 + 82 + 82 + 80 + 87 + 77}{3 \times 2 \times 2} = 77.33$$

Two-way ANOVA

Like one-way ANOVA, the two-way ANOVA is conducted by comparing various sum of squares. Recall that for one-way ANOVA we had

$$SS_T = SS_B + SS_W$$

where SS_B was the between groups sum of squares and SS_W was the within groups sum of squares.

Two-way ANOVA

For two-way ANOVA, the **total variability** can be described by the total sum of squares

$$SS_T = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(Y_{ijk} - \overline{Y}_{...})^2$$

and we will see that the total variability can be split up into the within-cell variability and the between-group variability, where

- The **within-cell variability** (often referred to as errors or residuals) is defined by SS_E (analog of SS_W)

$$SS_E = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(Y_{ijk} - \overline{Y}_{ij.})^2$$

Two-way ANOVA

- The **between-cell variability** can be split up into three components (analog of SS_B)
  - SS_A: the sum of squares for the row factor

$$SS_A = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(\overline{Y}_{i..} - \overline{Y}_{...})^2 = JK\sum_{i=1}^{I}(\overline{Y}_{i..} - \overline{Y}_{...})^2$$

  - SS_B: the sum of squares for the column factor

$$SS_B = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(\overline{Y}_{.j.} - \overline{Y}_{...})^2 = IK\sum_{j=1}^{J}(\overline{Y}_{.j.} - \overline{Y}_{...})^2$$

  - SS_{AB}: the sum of squares for the interaction between the row and column factor

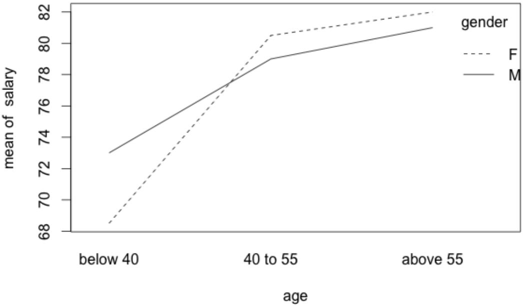
$$\begin{aligned} SS_{AB} &= \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(\overline{Y}_{ij.} - \overline{Y}_{i..} - \overline{Y}_{.j.} + \overline{Y}_{...})^2 \\ &= K\sum_{i=1}^{I}\sum_{j=1}^{J}(\overline{Y}_{ij.} - \overline{Y}_{i..} - \overline{Y}_{.j.} + \overline{Y}_{...})^2 \end{aligned}$$

Two-way ANOVA

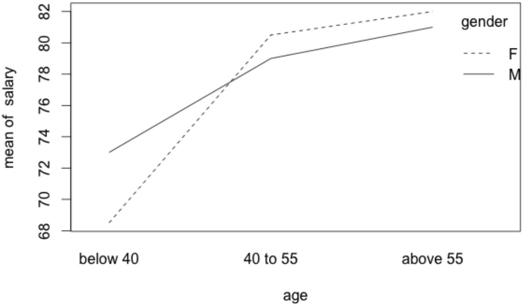
One way to graphically determine if there is an interaction effect is to use an **interaction plot**.

- y-axis is the response level
- x-axis is the levels of one of the factors
- There is a different curve for each level of the other factor.

If the curves appear to be more or less parallel, then we conclude that there is unlikely to be an interaction effect.



Two-way ANOVA



There appears to be no interaction for the older age groups with gender, but a bit of an interaction within the younger age groups with gender. We will see that this is not enough to conclude that there is a significant interaction over all levels.

Two-way ANOVA

The sum of squares satisfy

$$SS_T = SS_A + SS_B + SS_{AB} + SS_E$$

and if we assume that $\epsilon_{ijk} \sim N(0, \sigma^2)$, then

- $SS_E/\sigma^2 \sim \chi^2_{IJ(K-1)}$
- $SS_A/\sigma^2 \sim \chi^2_{I-1}$
- $SS_B/\sigma^2 \sim \chi^2_{J-1}$
- $SS_{AB}/\sigma^2 \sim \chi^2_{(I-1)(J-1)}$

Two-way ANOVA

The two-way ANOVA table is thus given by

Source	df	SS	MS	F
A (between)	$I-1$	SS_A	MS_A	MS_A/MS_E
B (between)	$J-1$	SS_B	MS_B	MS_B/MS_E
AB (between)	$(I-1)(J-1)$	SS_{AB}	MS_{AB}	MS_{AB}/MS_E
Error (within)	$IJ(K-1)$	SS_E	MS_E	
Total	$IJK-1$	SS_T		

where

$$MS = \frac{SS}{df}$$

Two-way ANOVA

The p-value for testing the null hypothesis that there is **no difference between the levels of factor** A is given by

$$P(F_{I-1,IJ(K-1)} \geq MS_A/MS_E)$$

The p-value for testing the null hypothesis that there is **no difference between the levels of factor** B is given by

$$P(F_{J-1,IJ(K-1)} \geq MS_B/MS_E)$$

The p-value for testing the null hypothesis that there is **no interaction between** A **and** B is given by

$$P(F_{(I-1)(J-1),IJ(K-1)} \geq MS_{AB}/MS_E)$$

Two-way ANOVA

For our example,

	Male	Female
age less than 40	71, 75	67, 70
age 40 to 55	77, 81	79, 82
age above 55	82, 80	87, 77

$$SS_{Age} = 266.17$$

$$SS_{Gender} = 1.33$$

$$SS_{Age \times Gender} = 22.17$$

$$SS_{Error} = 77.00$$

Two-way ANOVA

For our example:

Source	df	SS	MS	F
Gender	1	1.13	1.13	0.10
Age	2	226.17	133.08	10.37
Age $\times$ Gender	2	22.17	11.08	0.86
Error	6	77.00	12.83	
Total	11	366.67		

and our p-values are:

$$\text{Gender effect: } P(F_{1,6} \geq 0.10) = 0.76$$
$$\text{Age effect: } P(F_{2,6} \geq 10.37) = 0.011$$
$$\text{Interaction effect: } P(F_{2,6} \geq 0.86) = 0.47$$

So we conclude that there is a difference in salary between the different age groups but not for the different genders. There also does not appear to be a significant interaction effect.

Exercise

Exercise: Two-way ANOVA

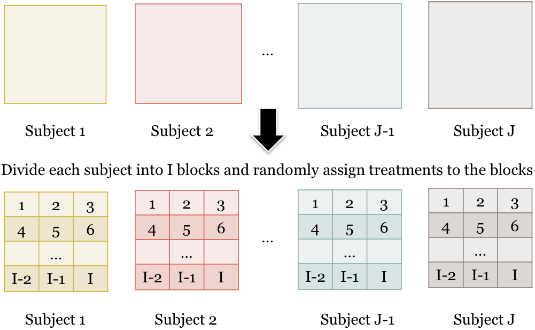
Do the salary example in R, and report your results in a .pdf format. Options include (but are not limited to):

- Using knitr (.Rnw file) – my preferred method (allows for easy incorporation of LaTeX, `R` code and figures)
- Using R Markdown (.Rmd file)
- Using IPython notebook

Randomized Block Design

Randomized Block Design

Suppose we have I treatments (e.g. fertilizers, diets) that we want to try on each of I subjects (e.g. plots of land, people).



The I blocks for a subject might be

- physical partitions of the plot of land where a different fertilizer is applied to each of the blocks
- stretches of time in each of which the same subject is put on a different diet.

Randomized Block Design

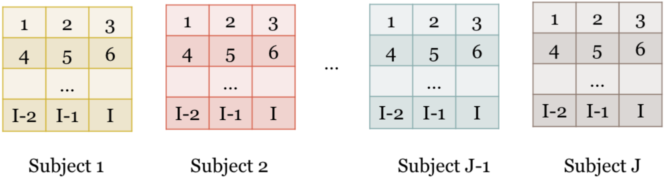
How is this different from one-way ANOVA?

- One-way ANOVA: we have a different group of J subjects for each of the I treatments
- Randomized Block Design: we have the same group of J subjects for each of the I treatments

How is this different from two-way ANOVA?

- Two-way ANOVA: Subjects are considered replicates within factors
- Randomized Block Design: Subjects are themselves a factor

Randomized Block Design



The observation in the (i, j)th block, Y_{ij}, is modeled by

$$Y_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij}$$

- μ is the overall mean parameter
- α_i is the differential effect of the ith treatment.
  - Assume $\sum_{i=1}^{I} \alpha_i = 0$.
- β_i is the differential effect of the jth subject.
  - Assume $\sum_{j=1}^{J} \beta_j = 0$.
- ϵ_{ij} is the random error for the ith treatment on the jth subject.
  - Assume $\epsilon_{ij} \overset{IID}{\sim} N(0, \sigma^2)$.

Randomized Block Design

The total amount of variation is given by:

$$SS_T = \sum_{i=1}^{I} \sum_{j=1}^{J} (Y_{ij} - \overline{Y})^2$$

The variation explained by the treatment differential effect is given by:

$$SS_A = J \sum_{i=1}^{I} (\overline{Y}_{i\cdot} - \overline{Y})^2$$

The variation explained by the subject differential effect is given by:

$$SS_B = I \sum_{j=1}^{J} (\overline{Y}_{\cdot j} - \overline{Y})^2$$

The variation **not explained by the model** is given by:

$$SS_{AB} = \sum_{i=1}^{I} \sum_{j=1}^{J} (Y_{ij} - \overline{Y}_{i\cdot} - \overline{Y}_{\cdot j} + \overline{Y})^2$$

Randomized Block Design

As usual, we have

$$SS_T = SS_A + SS_B + SS_{AB}$$

and that

$$MS_A = \frac{SS_A}{I-1} \quad MS_B = \frac{SS_B}{J-1} \quad MS_{AB} = \frac{SS_{AB}}{(I-1)(J-1)}$$

To test the null hypothesis that there is no treatment effect:

$$H_0 : \alpha_i = 0 \quad \forall i$$

our p-value is given by

$$\boxed{P\left(F_{I-1,(I-1)(J-1)} \geq \frac{MS_A}{MS_{AB}}\right)}$$

Exercise

Exercise: Randomized block design (Rice 12.3 Example A)

Let's consider an experimental study of drugs to relieve itching.

- 5 drugs were compared to a placebo and no drug (7 treatments).
- 10 volunteers male subjects aged 20-30.
- Each volunteer underwent one treatment per day, and the time-order was randomized.
- Individuals (rather than treatments) are the "blocks".
- The subjects were given a drug (or placebo) intravenously, and itching was induced on their forearms.
- The subjects recorded the duration of the itching.

Exercise: Randomized block design (Rice 12.3 Example A)

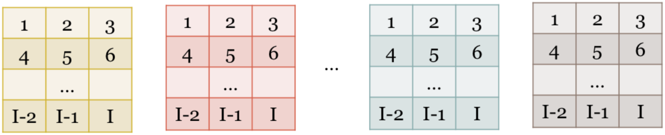
The following table recorded the durations of the itching (in seconds) for the first 5 subjects.

Subject	No Drug	Placebo	Papa.	Morph.	Amino.	Pento.	Tripelen.
BG	174	263	105	199	141	108	141
JF	224	213	103	143	168	341	184
BS	260	231	145	113	78	159	125
SI	225	291	103	225	164	135	227
BW	165	168	144	176	127	239	194

Test the null hypothesis that there is no difference in means between the different treatments.

Friedman's Test (non-parametric version of the Randomized block design)

Friedman's Test



Subject 1   Subject 2   Subject J-1   Subject J

1. Calculate the ranks of each treatment **within each of the J blocks** (rather than overall). R_{ij} is the rank of the ith treatment for the jth subject *for the jth subject.*
2. Compute

$$Q = \frac{12}{I(I+1)} SS_A = \frac{12J}{I(I+1)} \sum_{i=1}^{I} \left(\overline{R}_{i\cdot} - \overline{R}\right)^2 \sim \chi^2_{I-1}$$

To test the null hypothesis that there is no treatment effect, the p-value is given by

$$P(\chi^2_{I-1} \geq Q)$$

Exercise

Exercise: Friedman's test (Rice 12.4 Example A)

Do the previous example using a non-parametric method.